Experimental Mathematics & Computer Algebra

Bruno Salvy

Part II

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Short Summary of Part I

Experimental Mathematics

A 3-step process:

1. Compute a high order approximation

   (high precision numerical approx.,
   power series truncated to high order,
   large number of terms in a sequence,…)

2. Guess/conjecture a general formula

   (with the help of a computer)

3. Prove it

   (using computer-algebra algorithms)

Basic Computer Algebra

Simplification is undecidable.

Fast computation with large integers, polynomials, power series, matrices,…

Tools for conjectures:

OEIS (integer sequences)
ISC (real numbers)
Padé-Hermite approximants (relations between power series)
LLL (relations between numerical approximations).

Exercises for Part I

Ex 1. Identities for Nice Constants

$$\sum_{i=0}^{\infty} \frac{16^{-i}}{8i+j}, \qquad j = 1, \ldots, 8$$

lead to formulas for

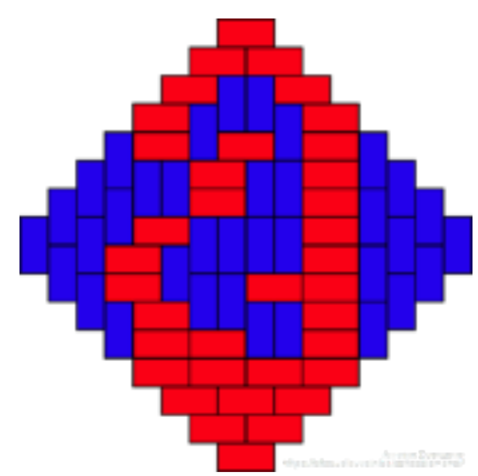
$$\pi, \ln 2, \ln 3, \ln 5, \arctan 2, \arctan 3, \sqrt{2}\arctan(1/\sqrt{2}), \sqrt{2}\ln(1+\sqrt{2}).$$

Ex 2. Stanley's Problem E 2297

Number of monomials in a generic *nxn* symmetric matrix

Ex 3. Number of Domino Tilings of the Aztec Diamond

Guess & Prove a formula



Part II: Tools for Proofs

A short tour of ``univariate'' computer algebra

1. Resultants
2. D-finite sequences & series
3. Creative telescoping

1. Resultants

Polynomials as a data-structure

Definition

The Sylvester matrix of $A = a_m x^m + \cdots + a_0 \in \mathbb{K}[x]$, $(a_m \neq 0)$, and of $B = b_n x^n + \cdots + b_0 \in \mathbb{K}[x]$, $(b_n \neq 0)$, is the square matrix of size $m+n$

$$\mathsf{Syl}(A,B) = \begin{bmatrix} a_m & a_{m-1} & \dots & a_0 & & & & \\ & a_m & a_{m-1} & \dots & a_0 & & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & a_m & a_{m-1} & & \dots & a_0 \\ b_n & b_{n-1} & \dots & b_0 & & & & \\ & b_n & b_{n-1} & \dots & b_0 & & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & b_n & b_{n-1} & & \dots & b_0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n \text{ rows} \\ \left.\vphantom{\begin{matrix}b\\b\\b\\b\end{matrix}}\right\} m \text{ rows} \end{matrix}$$

The resultant $\mathsf{Res}(A,B)$ of A and B is the determinant of $\mathsf{Syl}(A,B)$.

- Definition extends to polynomials with coefficients in a commutative ring R.

Basic Observation

If $\quad A = a_m x^m + \cdots + a_0 \quad$ and $\quad B = b_n x^n + \cdots + b_0, \quad$ then

$$\begin{bmatrix} a_m & a_{m-1} & \dots & a_0 & & & \\ & \ddots & \ddots & & & \ddots & \\ & & a_m & a_{m-1} & & \dots & a_0 \\ b_n & b_{n-1} & \dots & b_0 & & & \\ & \ddots & \ddots & & & \ddots & \\ & & b_n & b_{n-1} & & \dots & b_0 \end{bmatrix} \times \begin{bmatrix} \alpha^{m+n-1} \\ \vdots \\ \alpha \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha^{n-1} A(\alpha) \\ \vdots \\ A(\alpha) \\ \alpha^{m-1} B(\alpha) \\ \vdots \\ B(\alpha) \end{bmatrix}$$

Corollary: If $A(\alpha) = B(\alpha) = 0$, then $\mathsf{Res}\,(A, B) = 0$.

Example: the Discriminant

The discriminant of A is the resultant of A and of its derivative A'.

E.g. for $A = ax^2 + bx + c$,

$$\mathsf{Disc}(A) = \mathsf{Res}\,(A, A') = \det \begin{bmatrix} a & b & c \\ 2a & b & \\ & 2a & b \end{bmatrix} = -a(b^2 - 4ac).$$

E.g. for $A = ax^3 + bx + c$,

$$\mathsf{Disc}(A) = \mathsf{Res}\,(A, A') = \det \begin{bmatrix} a & 0 & b & c & \\ & a & 0 & b & c \\ 3a & 0 & b & & \\ & 3a & 0 & b & \\ & & 3a & 0 & b \end{bmatrix} = a^2(4b^3 + 27ac^2).$$

- The discriminant vanishes when A and A' have a common root, that is when A has a multiple root.

Main Properties

- Link with gcd $\text{Res}\,(A,B) = 0$ if and only if $\gcd(A,B)$ is non-constant.

- Elimination property
  There exist $U, V \in \mathbb{K}[x]$ not both zero, with $\deg(U) < n$, $\deg(V) < m$ and such that the following Bézout identity holds:

$$\text{Res}\,(A,B) = UA + VB \quad \text{in} \;\; \mathbb{K} \cap (A,B).$$

- Poisson formula
  If $A = a(x-\alpha_1)\cdots(x-\alpha_m)$ and $B = b(x-\beta_1)\cdots(x-\beta_n)$, then

$$\text{Res}\,(A,B) \;=\; a^n b^m \prod_{i,j}(\alpha_i - \beta_j) \;=\; a^n \prod_{1\leq i\leq m} B(\alpha_i).$$

- Bézout-Hadamard bound
  If $A, B \in \mathbb{K}[x,y]$, then $\text{Res}_y(A,B)$ is a polynomial in $\mathbb{K}[x]$ of degree

$$\leq \deg_x(A)\deg_y(B) + \deg_x(B)\deg_y(A).$$

Application: Computation with Algebraic Numbers

Let $A = \prod_i (x - \alpha_i)$ and $B = \prod_j (x - \beta_j)$ be polynomials of $\mathbb{K}[x]$. Then

$$\mathsf{Res}_x(A(x), B(t-x)) = \prod_{i,j} (t - (\alpha_i + \beta_j)),$$

$$\mathsf{Res}_x(A(x), B(t+x)) = \prod_{i,j} (t - (\beta_j - \alpha_i)),$$

$$\mathsf{Res}_x(A(x), x^{\deg B} B(t/x)) = \prod_{i,j} (t - \alpha_i \beta_j),$$

$$\mathsf{Res}_x(A(x), t - B(x)) = \prod_i (t - B(\alpha_i)).$$

In particular, the set of algebraic numbers is a field.

Proof: Poisson's formula. E.g., first one: $\prod_i B(t - \alpha_i) = \prod_{i,j} (t - \alpha_i - \beta_j)$.

▶ The same formulas apply mutatis mutandis to algebraic power series.

Exercise for the afternoon: A Nice Number

Guess and prove a simple formula for

$$\frac{\sin\frac{2\pi}{7}}{\sin^2\frac{3\pi}{7}} - \frac{\sin\frac{\pi}{7}}{\sin^2\frac{2\pi}{7}} + \frac{\sin\frac{3\pi}{7}}{\sin^2\frac{\pi}{7}}.$$

Rothstein-Trager Resultant

Let $A, B \in \mathbb{K}[x]$ with $\deg(A) < \deg(B)$ and squarefree monic denominator B. The rational function $F = A/B$ has simple poles only.

If $F = \sum_i \frac{\gamma_i}{x - \beta_i}$, then the residue γ_i of F at the pole β_i equals $\gamma_i = \frac{A(\beta_i)}{B'(\beta_i)}$.

Theorem. The residues γ_i of F are roots of the Rothstein-Trager resultant

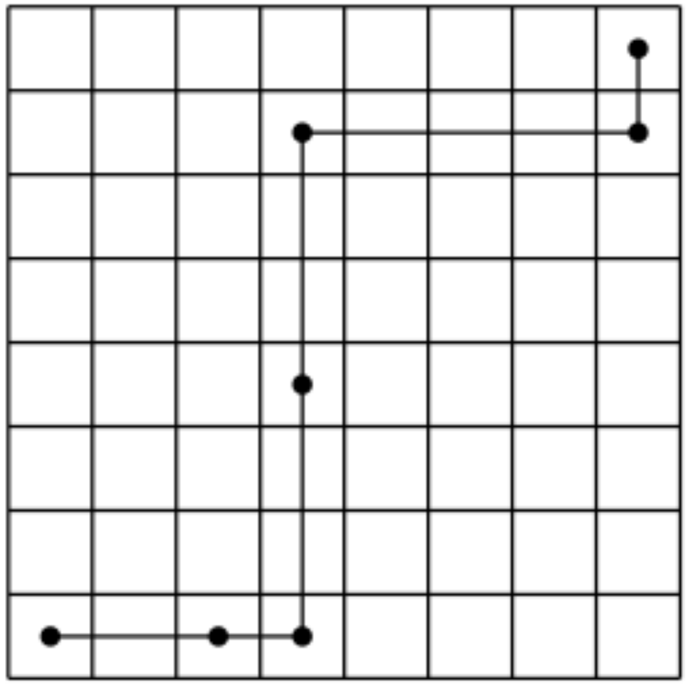
$$R(t) = \mathsf{Res}_x\big(B(x),\ A(x) - t \cdot B'(x)\big).$$

Proof. Poisson formula again: $R(t) = \prod_i \Big(A(\beta_i) - t \cdot B'(\beta_i)\Big)$.

- This special resultant is useful for symbolic integration of rational functions.

Application: Diagonal Rook Paths

Question: A chess Rook can move any number of squares horizontally or vertically in one step. How many paths can a Rook take from the lower-left corner square to the upper-right corner square of an $N \times N$ chessboard? Assume that the Rook moves right or up at each step.



$$1,\ 2,\ 14,\ 106,\ 838,\ 6802,\ 56190,\ 470010,\ \ldots$$

Diagonal Rook Paths II

$$1,\ 2,\ 14,\ 106,\ 838,\ 6802,\ 56190,\ 470010,\ \ldots$$

$$\mathsf{Diag}(F) = [s^0]\, F(s, x/s) = \frac{1}{2i\pi} \oint F(s, x/s)\, \frac{ds}{s}, \quad \text{where} \quad F = \frac{1}{1 - \frac{s}{1-s} - \frac{t}{1-t}}.$$

By the residue theorem, $\mathsf{Diag}(F)$ is a sum of roots of the Rothstein-Trager resultant

```
> F:=1/(1-s/(1-s)-t/(1-t)):
> G:=normal(1/s*subs(t=x/s,F)):
> factor(resultant(denom(G),numer(G)-t*diff(denom(G),s),s));
```

$$x^2\,(2\,t-1)\,(x-1)\,\left(36\,t^2\,x-4\,t^2-x+1\right)$$

Answer: Generating series of diagonal Rook paths is $\displaystyle \frac{1}{2}\left(1+\sqrt{\frac{1-x}{1-9x}}\right).$

Application: Certified Algebraic Guessing

Guess+Bound=Proof

Theorem. Suppose $A \in \mathbb{K}[[x]]$ is an algebraic series, and that it is a root of a (unknown) polynomial in $\mathbb{K}[x, y]$ of degree at most d in x and at most n in y.

If $\displaystyle\sum_{i=0}^{n} Q_i(x) A^i(x) = O(x^{2dn})$, then $\displaystyle\sum_{i=0}^{n} Q_i(x) A^i(x) = 0$.

Proof: Let $P \in \mathbb{K}[x, y]$ be an irreducible polynomial such that

$$P(x, A(x)) = 0, \text{ and } \deg_x(P) \leq d, \deg_y(P) \leq n.$$

- By Hadamard, $R(x) = \mathrm{Res}_y(P, Q) \in \mathbb{K}[x]$ has degree at most $2dn$.
- By elimination, $R(x) = UP + VQ$ for $U, V \in \mathbb{K}[x, y]$ with $\deg_y(V) < n$.
- Evaluation at $y = A(x)$ yields

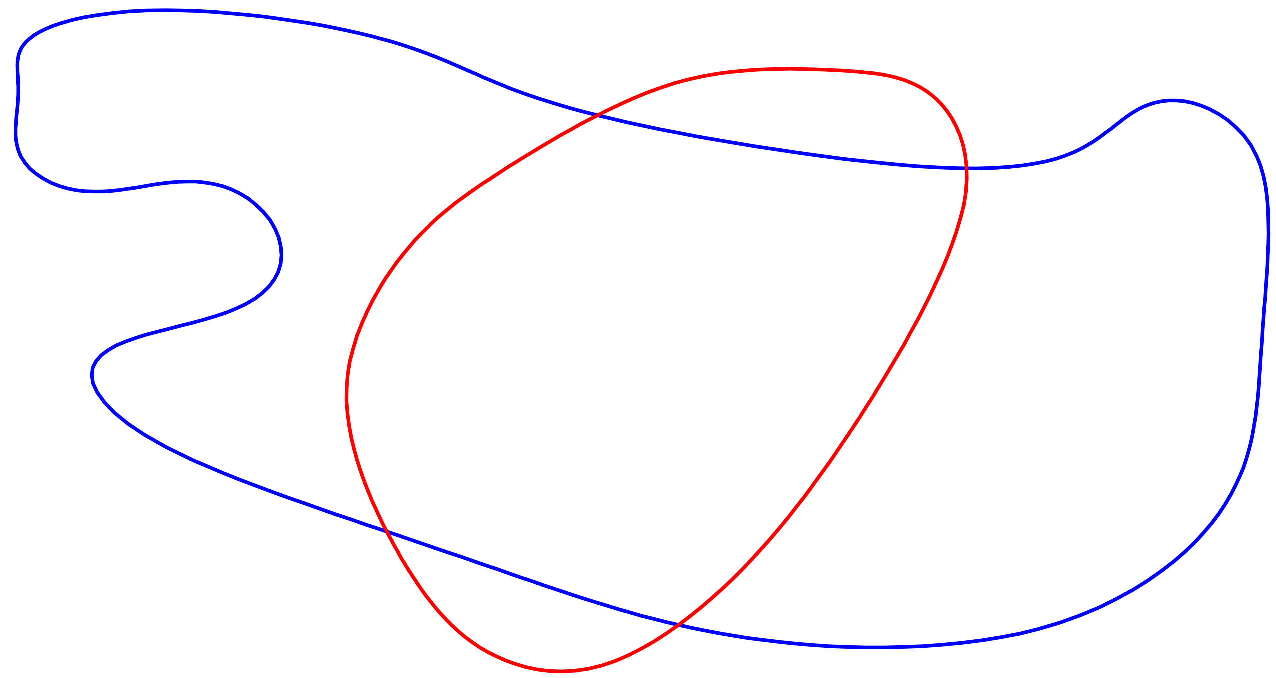
$$R(x) = U(x, A(x)) \underbrace{P(x, A(x))}_{0} + V(x, A(x)) \underbrace{Q(x, A(x))}_{O(x^{2dn})} = O(x^{2dn}).$$

- Thus $R = 0$, that is $\gcd(P, Q) \neq 1$, and thus $P \mid Q$, and A is a root of Q.

Systems of 2 Equations in 2 Unknowns

Geometrically, roots of a polynomial $f \in \mathbb{Q}[x]$ correspond to points on a line.

Roots of polynomials $A \in \mathbb{Q}[x, y]$ correspond to plane curves $A = 0$.

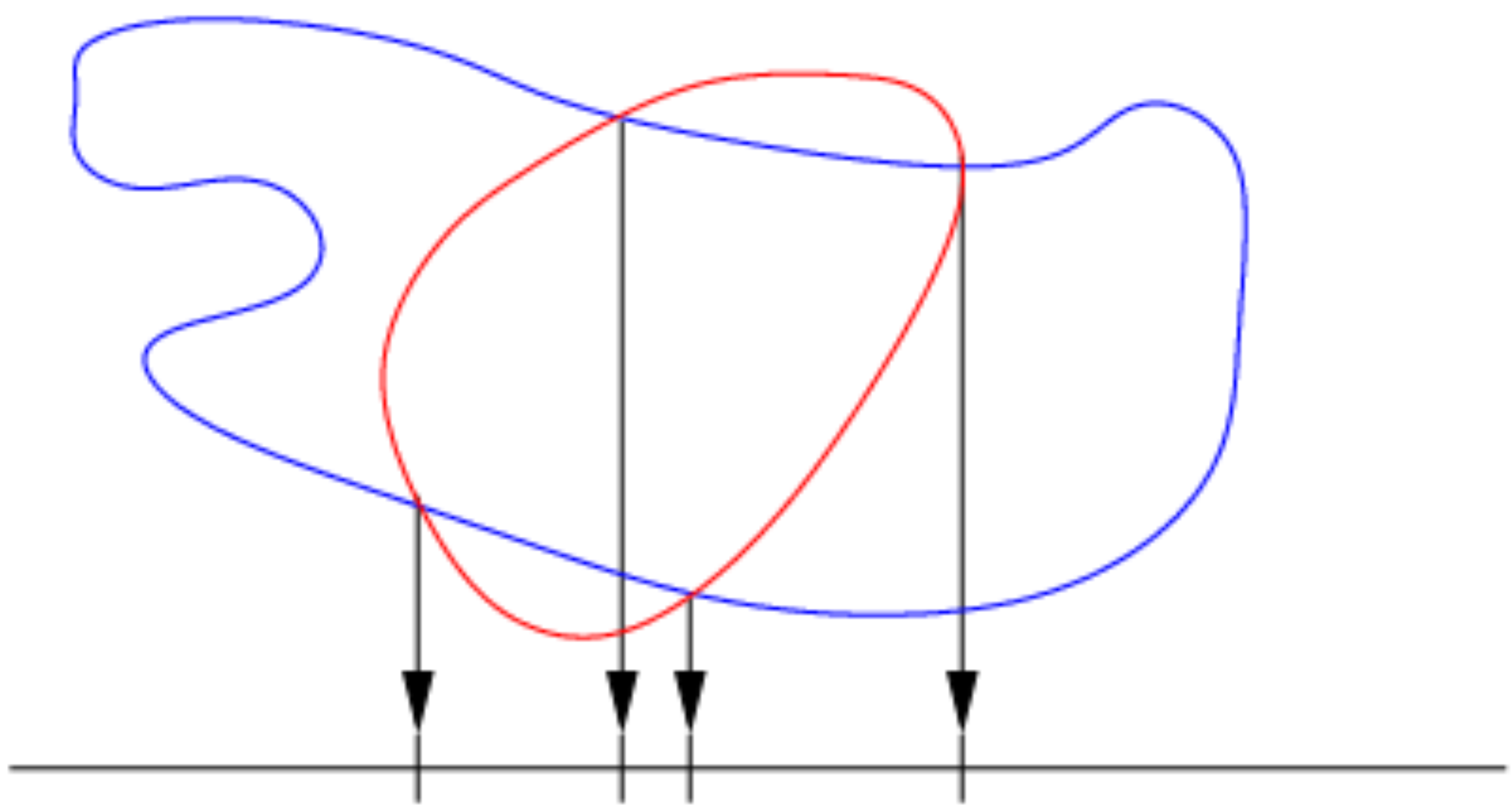


Let now A and B be in $\mathbb{Q}[x, y]$. Then:

- either the curves $A = 0$ and $B = 0$ have a common component,
- or they intersect in a finite number of points.

Resultants Compute Projections

Theorem. Let $A = a_m y^m + \cdots$ and $B = b_n y^n + \cdots$ be polynomials in $\mathbb{Q}[x][y]$. The roots of $\mathsf{Res}_y(A, B) \in \mathbb{Q}[x]$ are either the abscissas of points in the intersection $A = B = 0$, or common roots of a_m and b_n.



Proof. Elimination property: $\mathsf{Res}(A, B) = UA + VB$.

Thus $A(\alpha, \beta) = B(\alpha, \beta) = 0$ implies $\mathsf{Res}_y(A, B)(\alpha) = 0$

Application: Implicitization of Parametric Curves

Task: Given a rational parametrization of a curve

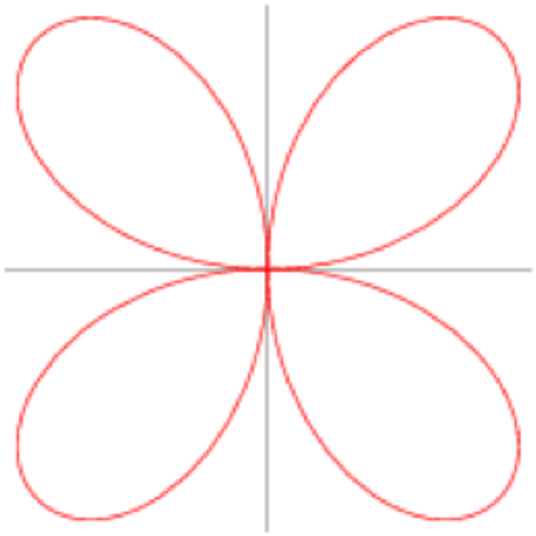
$$x = A(t), \quad y = B(t), \qquad A, B \in \mathbb{K}(t),$$

compute a non-trivial polynomial in x and y vanishing on the curve.

Recipe: take the resultant in t of numerators of $x - A(t)$ and $y - B(t)$.

Example: for the four-leaved clover (a.k.a. quadrifolium) given by

$$x = \frac{4t(1-t^2)^2}{(1+t^2)^3}, \quad y = \frac{8t^2(1-t^2)}{(1+t^2)^3},$$



$$\mathrm{Res}_t((1+t^2)^3x - 4t(1-t^2)^2, (1+t^2)^3y - 8t^2(1-t^2)) = 2^{24}\left((x^2+y^2)^3 - 4x^2y^2\right).$$

2. D-finite Series and Sequences

Differential or Recurrence equations as a data-structure

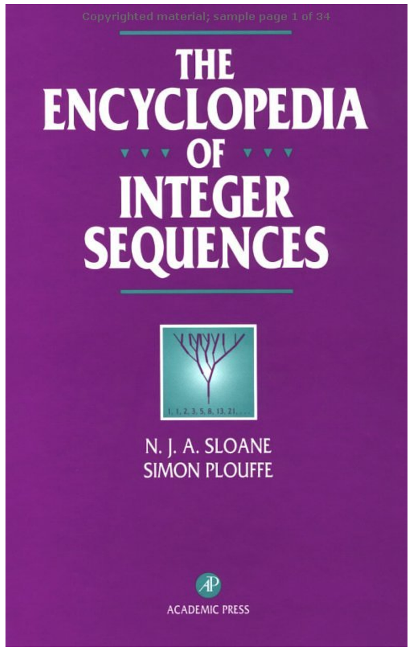
D-finite Series & Sequences

Definition: A power series $f(x) \in \mathbb{K}[[x]]$ is D-finite over $\mathbb{K}$ when its derivatives generate a finite-dimensional vector space over $\mathbb{K}(x)$.

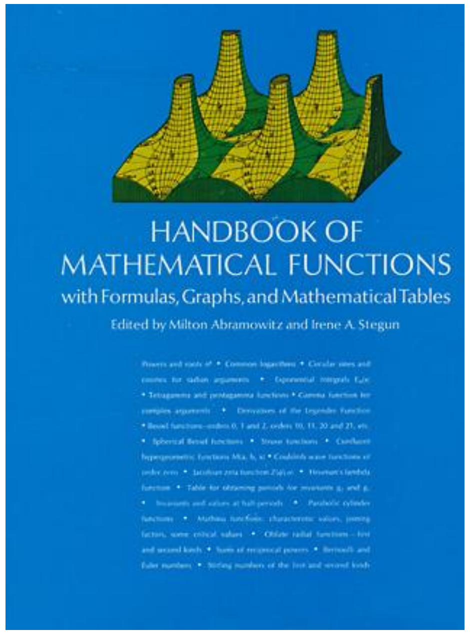
A sequence u_n is D-finite (or P-recursive) over $\mathbb{K}$ when its shifts $(u_n, u_{n+1}, \dots)$ generate a finite-dimensional vector space over $\mathbb{K}(n)$.

equation + init conditions = data structure

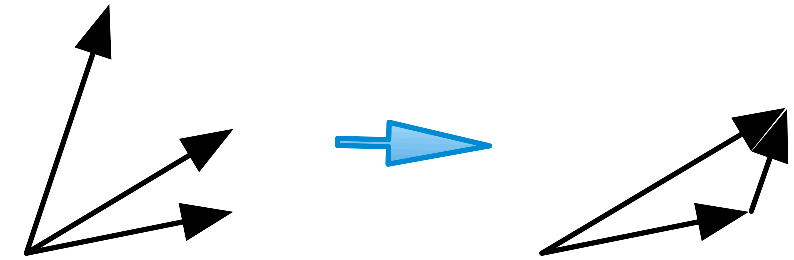
About 25% of Sloane's encyclopedia, 60% of Abramowitz & Stegun



Examples: exp, log, sin, cos, sinh, cosh, arccos, arccosh, arcsin, arcsinh, arctan, arctanh, arccot, arccoth, arccsc, arccsch, arcsec, arcsech, ${}_pF_q$ (includes Bessel J, Y, I and K, Airy Ai and Bi and polylogarithms), Struve, Weber and Anger functions, the large class of algebraic functions,...



Automatic Proof of Identities



```
> series(sin(x)^2+cos(x)^2-1,x,4);
```

$O(x^4)$

f satisfies a LDE
$\iff$
f,f′,f″,… live in a finite-dim. vector space

Why is this a proof?

1. sin and cos satisfy a 2nd order LDE: $y''+y=0$;
2. their squares and their sum satisfy a 3rd order LDE;
3. the constant -1 satisfies $y'=0$;
4. thus $\sin^2+\cos^2-1$ satisfies a LDE of order at most 4;
5. the Cauchy-Lipschitz theorem concludes.

Proofs of non-linear identities by linear algebra!

Mehler's identity for Hermite polynomials

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$

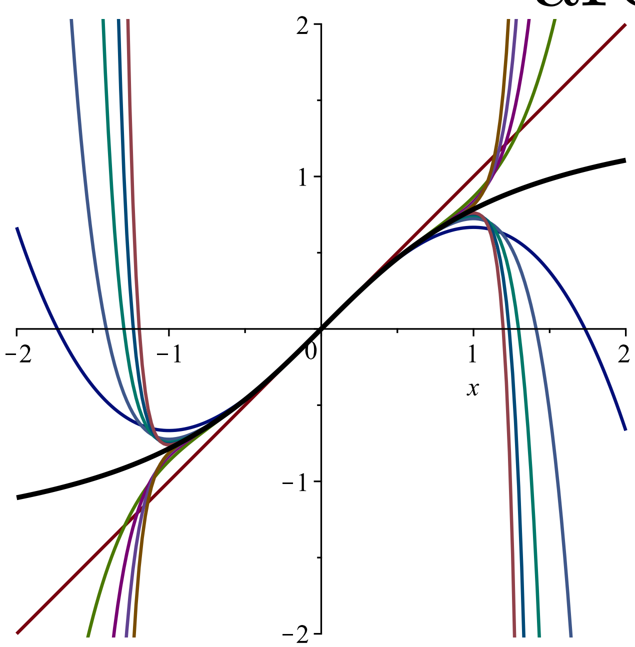
1. Definition of Hermite polynomials:
   recurrence of order 2;
2. Product by linear algebra: $H_{n+k}(x)H_{n+k}(y)/(n+k)!$, $k\in\mathbb{N}$
   generated over $\mathbb{Q}(x,n)$ by

   $$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$

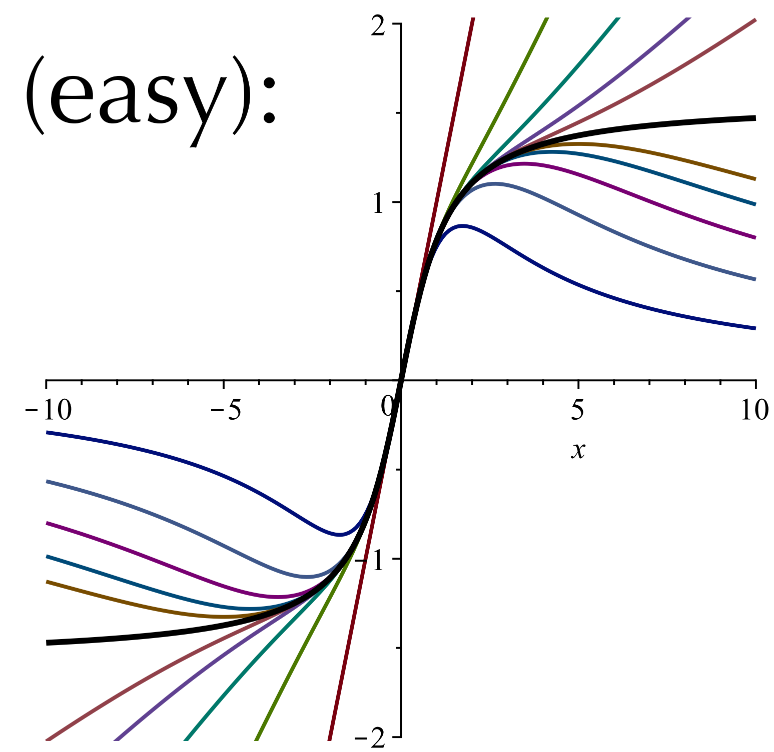
   → recurrence of order at most 4;
3. Translate into differential equation.

Guess & prove continued fractions

1. Taylor expansion produces first terms (easy):



$$\arctan x = \cfrac{x}{1 + \cfrac{\frac{1}{3}x^2}{1 + \cfrac{\frac{4}{15}x^2}{1 + \cfrac{\frac{9}{35}x^2}{1 + \cdots}}}}$$



2. Guess a formula (easy): $a_n = \dfrac{n^2}{4n^2 - 1}$

3. Prove that the CF with these a_n converges to arctan.

No human intervention needed.

`gfun[ContFrac]`

Automatic Proof of the Guessed CF (1/2)

$$\arctan x \stackrel{?}{=} \frac{x|}{|1} + \cdots + \frac{\frac{n^2}{4n^2-1}x^2|}{|1} + \cdots$$

Lemma. Let P_n/Q_n be the nth convergent. Then

$$\lim_{n\to\infty}\left((x^2+1)\left(\frac{P_n}{Q_n}\right)' - 1\right) = 0 \Rightarrow \lim_{n\to\infty}\frac{P_n}{Q_n} = \arctan x.$$

Let $H_n := Q_n^2((x^2+1)(P_n/Q_n)' - 1).$

1. Compute a linear recurrence for H_n;
2. Using the initial conditions, find a smaller-order one;
3. Conclude that $H_n = O(x^n)$.

Automatic Proof of the Guessed CF (2/2)

$$H_n := Q_n^2((x^2+1)(P_n/Q_n)' - 1).$$

1. P_n and Q_n satisfy $u_n = u_{n-1} + a_n x^2 u_{n-2}$ and $Q_n(0) \neq 0$.
2. H_n is a polynomial in P_n, Q_n and their derivatives.
3. All its shifts H_{n+k} are linear combinations of

$$P'_{n+i}Q_{n+j}, P_{n+i}Q'_{n+j}, P_{n+i}P_{n+j}, Q_{n+i}Q_{n+j}, \quad i \text{ and } j \text{ in } \{0,1\}$$

4. → by linear algebra

$$H_{n+4} = H_{n+3} + (\cdots x^2 + \cdots x^4)H_{n+2} + \cdots x^4 H_{n+1} + \cdots x^8 H_n$$

5. Using the initial conditions gives

$$(2n+3)^2 H_{n+1} + (n+2)^2 x^2 H_n = 0. \qquad \square$$

All these steps are easy to automate.

Algebraic Series can be Computed Fast

$$P(X, Y(X)) = 0 \qquad P \text{ irreducible}$$

Wanted: the first *N* Taylor coefficients of *Y*.

$$P_x(X, Y(X)) + P_y(X, Y(X)) \cdot Y'(X) = 0$$

$$Y'(X) = (-P_x P_y^{-1} \bmod P)(X, Y(X))$$

a polynomial

Note:
F sol LDE
⇒ *F(Y(X))* sol LDE
(same argument)

$$Y(X), Y'(X), Y''(X), \ldots \text{ in } \mathrm{Vect}_{\mathbb{Q}(X)}(1, Y, Y^2, \ldots)$$

finite dimension

→ a LDE by linear algebra

[Abel1827,Cockle1861,Harley1862,Tannery1875]

An Olympiad Problem

Question: Let (a_n) be the sequence with $a_0 = a_1 = 1$ satisfying the recurrence

$$(n+3)a_{n+1} = (2n+3)a_n + 3na_{n-1}.$$

Show that all a_n is an integer for all n.

Computer-aided solution: Let's compute the first 10 terms of the sequence:

```
> rec:=(n+3)*a(n+1)-(2*n+3)*a(n)-3*n*a(n-1): ini:=a(0)=1,a(1)=1:
> pro:=gfun:-rectoproc({rec,ini}, a(n), list);
> pro(10);
```

```
[1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188]
```

gfun's seriestoalgeq command allows to guess that GF is algebraic:

```
> pol:=gfun:-listtoalgeq(%,y(x))[1];
```

```
                              2     2
1 + (x - 1) y(x) + x  y(x)
```

Thus it is very likely that $y = \sum_{n\geq 0} a_n x^n$ verifies $1 + (x-1)y + x^2y^2 = 0$.

By coefficient extraction, (a_n) conjecturally verifies the non-linear recurrence

$$a_{n+2} = a_{n+1} + \sum_{k=0}^{n} a_k \cdot a_{n-k}. \tag{1}$$

Clearly (1) implies $a_n \in \mathbb{N}$. To prove (1), we proceed the other way around: we start with $P(x,y) = 1 + (x-1)y + x^2y^2$, and show that it admits a power series solution whose coefficients satisfy the same linear recurrence as (a_n):

```
> deq:=gfun:-algeqtodiffeq(pol,y(x)):
> recb:=gfun:-diffeqtorec(deq,y(x),b(n));
```

```
recb := {(3 + 3 n) b(n) + (2 n + 5) b(n + 1) + (-4 - n) b(n + 2),
                                                 b(0) = 1, b(1) = 1}
```

▶ In fact,

$$a_n = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k+2} \binom{n}{k} \binom{2k+2}{k+1} = \sum_{j\geq 0} (-1)^j \binom{n+1}{j} \binom{2n-3j}{n},$$

(which clearly implies $a_n \in \mathbb{Z}$) but how to find algorithmically such a formula?

3. Creative Telescoping

Examples I: hypergeometric summation

- $\displaystyle\sum_{k\in\mathbb{Z}} (-1)^k \binom{a+b}{a+k}\binom{a+c}{c+k}\binom{b+c}{b+k} = \frac{(a+b+c)!}{a!b!c!}$

- $\displaystyle A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies the recurrence [Apéry78]:

$$(n+1)^3 A_{n+1} = (34n^3 + 51n^2 + 27n + 5)A_n - n^3 A_{n-1}.$$

(*Neither Cohen nor I had been able to prove this in the intervening two months* [Van der Poorten]).

- $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \sum_{j=0}^{k} \binom{n}{k}^3$ [Strehl92]

Examples II: Integrals

- $\displaystyle\int_0^1 \frac{\cos(zu)}{\sqrt{1-u^2}}\,du = \int_1^{+\infty} \frac{\sin(zu)}{\sqrt{u^2-1}}\,du = \frac{\pi}{2} J_0(z);$
- $\displaystyle\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x)\,dx = -\frac{\ln(1-a^4)}{2\pi a^2}$ [Glasser-Montaldi94];
- $\displaystyle\frac{1}{2\pi i}\oint \frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\,dy = \frac{H_n(x)}{\lfloor n/2 \rfloor !}$ [Doetsch30].

Examples III: Diagonals

Definition If $f(x_1,\dots,x_k) = \sum_{i_1,i_2,\dots,i_k\geq 0} c_{i_1,\dots,i_k} x_1^{i_1}\cdots x_k^{i_k} \in \mathbb{K}[[x_1,\dots,x_k]]$, then its diagonal is $\operatorname{Diag}(f) = \sum_{n\geq 0} c_{n,\dots,n} x^n \in \mathbb{K}[[x]]$.

- Diagonal k-D rook paths: $\operatorname{Diag} \dfrac{1}{1-\frac{x_1}{1-x_1}-\cdots-\frac{x_k}{1-x_k}}$;
- Hadamard product: $F(x)\odot G(x) = \sum_n f_n g_n x^n = \operatorname{Diag}(F(x)G(y))$;
- Algebraic series [Furstenberg67]: if $P(x,S(x))=0$ and $P_y(0,0)\neq 0$ then

$$S(x) = \operatorname{Diag}\left(y^2\frac{P_y(xy,y)}{P(xy,y)}\right).$$

- Apéry's sequence [Dwork80]:

$$\sum A_n z^n = \operatorname{Diag}\frac{1}{(1-x_1)((1-x_2)(1-x_3)(1-x_4)(1-x_5)-x_1x_2x_3)}.$$

Theorem [Lipshitz88] The diagonal of a rational (or algebraic, or even D-finite) series is D-finite.

Summation by Creative Telescoping

$$I_n := \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

IF one knows Pascal's triangle:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} = 2\binom{n}{k} + \binom{n}{k-1} - \binom{n}{k},$$

then summing over k gives

$$I_{n+1} = 2I_n.$$

The initial condition $I_0 = 1$ concludes the proof.

Creative Telescoping for Sums

$$F_n = \sum_k u_{n,k} =?$$

IF one knows $A(n, S_n)$ and $B(n, k, S_n, S_k)$ s.t.

$$(A(n, S_n) + \Delta_k B(n, k, S_n, S_k)) \cdot u_{n,k} = 0$$

(where Δ_k is the difference operator, $\Delta_k \cdot v_{n,k} = v_{n,k+1} - v_{n,k}$), then the sum “telescopes”, leading to

$$A(n, S_n) \cdot F_n = 0.$$

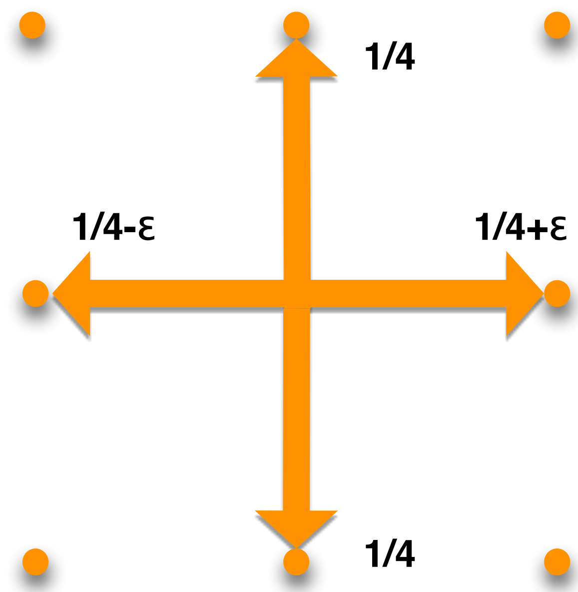
Zeilberger's Algorithm [1990]

Input: a hypergeometric term $u_{n,k}$, i.e., $u_{n+1,k}/u_{n,k}$ and $u_{n,k+1}/u_{n,k}$ rational functions in n and k;

Output:

- a linear recurrence (A) satisfied by $F_n = \sum_k u_{n,k}$
- a certificate (B), s.t. checking the result is easy from $A(n, S_n) \cdot u_{n,k} = \Delta_k B \cdot u_{n,k}$.

Example: SIAM flea



$$U_{n,k} := \binom{2n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k}\left(\frac{1}{4}+c\right)^k\left(\frac{1}{4}-c\right)^k\frac{1}{4^{2n-2k}}.$$

```
> SumTools[Hypergeometric][Zeilberger](U,n,k,Sn);
```

$$[(4\,n^2+16\,n+16)\,Sn^2+(-4\,n^2+32\,c^2n^2+96\,c^2n-12\,n+72\,c^2-9)\,Sn$$
$$+128\,c^4n+64\,c^4n^2+48\,c^4, \ldots\text{(BIG certificate)}\ldots]$$

Creative Telescoping for Integrals

$$I(x) = \int_\Omega u(x,y)\,dy = ?$$

IF one knows $A(x, \partial_x)$ and $B(x, y, \partial_x, \partial_y)$ s.t.

$$(A(x, \partial_x) + \partial_y B(x, y, \partial_x, \partial_y)) \cdot u(x,y) = 0,$$

then the integral "telescopes", leading to

$$A(x, \partial_x) \cdot I(x) = 0.$$

Special Case: Diagonals

Analytically,

$$\mathrm{Diag}(F(x,y)) = \frac{1}{2\pi i}\oint F\left(\frac{x}{y}, y\right)\frac{dy}{y}.$$

On power series,

$$(A(x,\partial_x) + \partial_y B)\cdot \underbrace{\frac{1}{y}F\left(\frac{x}{y}, y\right)}_{U} = 0 \Longrightarrow A(x,\partial_x)\cdot \mathrm{Diag}\, F = 0.$$

Proof:

1. $[y^{-1}]U = \mathrm{Diag}(f)$;
2. $[y^{-1}]A\cdot U + [y^{-1}]\partial_y B\cdot U = A\cdot [y^{-1}]U$.

Extends to more variables: $\mathrm{Diag}\, F(x,y,z)$ obtained from $[y^{-1}z^{-1}]U$, $U = \frac{1}{yz}F\left(\frac{x}{y}, \frac{y}{z}, z\right)$, **if** one finds

$$(A(x,\partial_x) + \partial_y B(x,y,z,\partial_x,\partial_y,\partial_z) + \partial_z C(x,y,z,\partial_x,\partial_y,\partial_z))\cdot U = 0.$$

Provided by Chyzak's algorithm

Summary of the Exercises for this Afternoon

4. A nice number

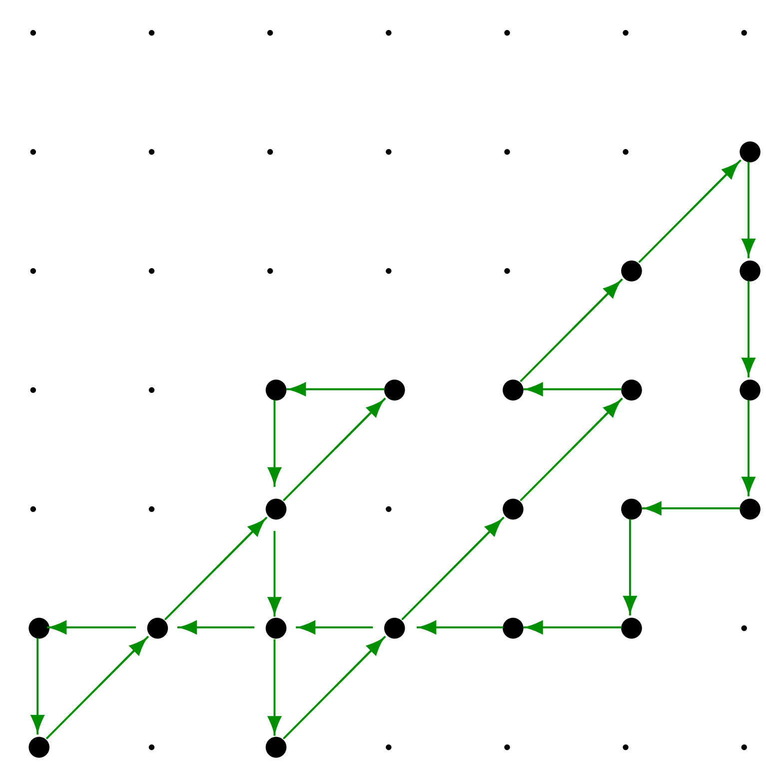
Guess and prove a simple formula for $\frac{\sin\frac{2\pi}{7}}{\sin^2\frac{3\pi}{7}} - \frac{\sin\frac{\pi}{7}}{\sin^2\frac{2\pi}{7}} + \frac{\sin\frac{3\pi}{7}}{\sin^2\frac{\pi}{7}}$.

See also the exercises on resultants.

Pb. Kreweras Excursions

$$K(t;x,y) = \sum_{n=0}^{\infty} \left(\sum_{i,j\geq 0} k(n;i,j) x^i y^j \right) t^n$$

is algebraic.



THE END

(Except for the exercises!)